ETMAG

LECTURE 7

- Properties of the limit of a function
- One-sided limits
- Asymptotes

Example 1. Find $\lim_{x\to 0} sin\frac{1}{x}$ if it exists.

We use Heine definition:

First, let $x_n = \frac{1}{2n\pi+\frac{\pi}{2}}$. Then, $\lim_{n\to\infty} sin\, x_n = \lim_{n\to\infty} sin\left(2n\pi + \frac{\pi}{2}\right) = \lim_{n\to\infty} 1 = 1.$

Next, use another sequence $y_n = \frac{1}{2n\pi-\frac{\pi}{2}}$. In this case, $\lim_{n\to\infty} sin\, y_n = \lim_{n\to\infty} sin\left(2n\pi - \frac{\pi}{2}\right) = \lim_{n\to\infty} (-1) = -1.$

We conclude that the function $sin\frac{1}{x}$ has no limit at 0 because we have found two sequences convergent to 0, (x_n) and (y_n), and the limits of corresponding sequences of values of f differ.

Example 2. Find $\lim_{x \to 0} x^2 \sin\frac{1}{x}$ or prove that it doesn't exists.

We will use Cauchy definition to show that $\lim_{x \to 0} x^2 \sin\frac{1}{x} = 0.$ Let ε be a positive real number. We must find a positive number δ such that $0 < |x - 0| < \delta$ implies $\left|x^2 \sin\frac{1}{x} - 0\right| < \varepsilon$. For every $x \neq 0$, $\left|x^2 \sin\frac{1}{x}\right| \leq \mathrm{x}^2$. So, if we put $\delta = \sqrt{\varepsilon}$, then whenever $0 < |x| < \delta = \sqrt{\varepsilon}$

$$\left|\mathrm{x}^2 \sin\frac{1}{\mathrm{x}}\right| \leq \mathrm{x}^2 < (\sqrt{\varepsilon})^2 = \varepsilon.$$

We are not bothered by the fact that $\sin\frac{1}{x}$ is undefined for x=0

FAQ.

1. In example 1, how to find such two sequences?
   There is no useful answer to this question other than the usual, boring one – practice, practice, practice …
2. How do I know whether I should try to show that the function has or that it does not have a limit at a point?
   The same boring answer.
3. Does example 2 indicate that we should try to find some sort of rule which assigns a *delta* to an *epsilon*?
   Yes, exactly. It does not have to be a function, though. Any δ such that $0 < \delta \leq \sqrt{\varepsilon}$ will do.
4. Sometimes you say *number* sometimes *point*. What is the difference?
   None, in this context. Every point on the real axis is a number and every number is a point on the real axis.

Limits involving infinity

1. Limits at infinity.

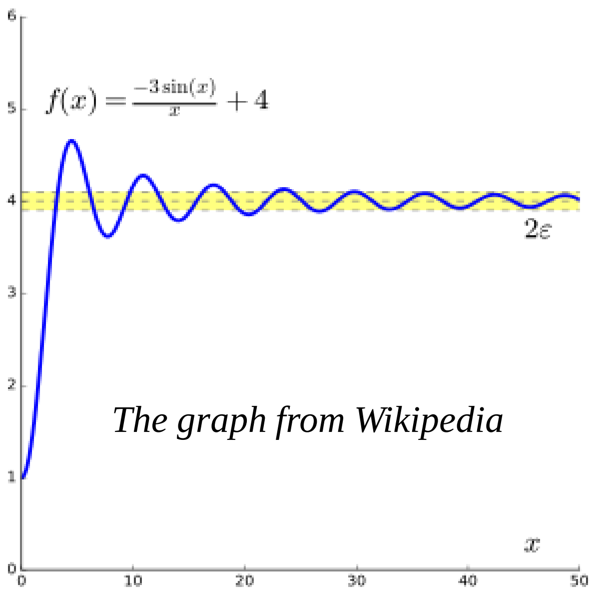
Definition.

A number L is the *limit of* f *as* x *approaches* ∞ iff

(C) $(\forall \varepsilon > 0)(\exists c \in \mathbb{R})(\forall x \in \mathbb{R})(x > c \Rightarrow |f(x) - L| < \varepsilon)$

(H) for every sequence (x_n), $\left(\lim_{n\to\infty} x_n = \infty \Rightarrow \lim_{n\to\infty} f(x_n) = L\right)$

In a similar way we define the limit of f as x approaches $-\infty$ (*minus infinity)*.



The graph from Wikipedia

We use notation

$$\lim_{x\to\infty} f(x) = L$$

or

$$\lim_{x\to-\infty} f(x) = L$$

Limits involving infinity

2. Infinity as the limit.

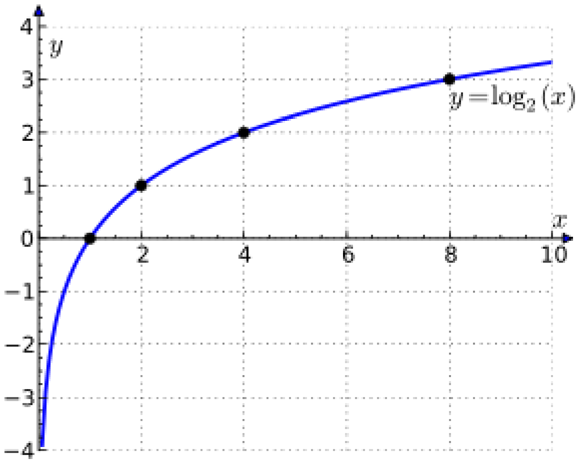
Definition.

For a function *f* and a point *c* we say $\lim_{x\to c} f(x) = \infty$ iff

(C) $(\forall \mathrm{M} \in \mathbb{R})(\exists \delta > 0)(\forall \mathrm{x} \in Dom(f))(0 < |\mathrm{x} - \mathrm{c}| < \delta \Rightarrow f(x) > M)$

(H) for every sequence (x_n), $\lim_{n\to\infty} x_n = c \Rightarrow \lim_{n\to\infty} f(x_n) = \infty$

In a similar way we define $\lim_{x\to c} f(x) = -\infty$



The graph from Wikipedia.

$\lim_{x\to 0} \log_2 x = -\infty$

Notice that we ignore points to the left of 0 – they do not belong to Dom_f.

Definition.
A number L is the *right-sided limit* of a function f at a point p iff L is the limit of the function $f^+ = f|_{(p;\infty)}$ at p.

A number L is the *left-sided limit* of a function f at a point p iff L is the limit of the function $f^- = f|_{(-\infty;p)}$ at p.

Right-sided and left-sided limits are denoted by $\lim_{x\to p^+} f(x)$ and $\lim_{x\to p^-} f(x)$, respectively.

In their campfire talks mathematicians use phrases like "*limit as x approaches p from the right*" or "*as x approaches p from above*".

All this boils down to "we ignore what happens to the right (or left) of p".

Example.

Let $f(x) = \frac{|x|}{x}$. The domain of f is $\mathbb{R} \setminus \{0\}$. Find $\lim_{x \to 0^+} f(x)$, $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0} f(x)$.

To find $\lim_{x \to 0^+} \frac{|x|}{x}$ we "let x approach 0 from above", which means we ignore what happens if $x \leq 0$. For positive x-s $\frac{|x|}{x} = \frac{x}{x} = 1,$ hence our limit $\lim_{x \to 0^+} \frac{|x|}{x} = 1$. Similarly, for x-s smaller than 0, $\frac{|x|}{x} = -1$, hence $\lim_{x \to 0^-} \frac{|x|}{x} = -1$.

Consider sequences $\left(\frac{1}{n}\right)$ and $\left(-\frac{1}{n}\right)$, both convergent to 0. Clearly, $\lim_{n \to \infty} f\left(\frac{1}{n}\right) = 1$ and $\lim_{n \to \infty} f\left(\frac{-1}{n}\right) = -1$. From Heine definition we conclude that $\lim_{x \to 0} f(x)$ doesn't exist.

The last example can be generalized into the following theorem:

Theorem.

The limit $\lim_{x\to p} f(x)$ exists if and only if $\lim_{x\to p^+} f(x)$ and $\lim_{x\to p^-} f(x)$ exist, and

$$\lim_{x\to p^+} f(x) = \lim_{x\to p^-} f(x)$$

Remark.
The idea of one-sided limits does not apply to limits at ∞ or $-\infty$.

It does apply, though, including the above theorem, to infinite limits.

Example.

$\lim_{x\to 0} \frac{1}{x}$ does not exist because $\lim_{x\to 0^+} \frac{1}{x} = \infty$ while $\lim_{x\to 0^-} \frac{1}{x} = -\infty$. On the other hand, $\lim_{x\to 0} \frac{1}{x^2} = \infty$ because $\lim_{x\to 0^+} \frac{1}{x^2} = \infty$ and $\lim_{x\to 0^-} \frac{1}{x^2} = \infty$

Theorem. (Properties of the limit)

Suppose that $\lim_{x\to p} f(x)$ and $\lim_{x\to p} g(x)$ exist. Then

1. $\lim_{x\to p}(f+g)(x) = \lim_{x\to p} f(x) + \lim_{x\to p} g(x)$,
2. $\lim_{x\to p}(f\cdot g)(x) = \lim_{x\to p} f(x) \cdot \lim_{x\to p} g(x)$,
3. $\lim_{x\to p}\left(\frac{f}{g}\right)(x) = \frac{\lim_{x\to p} f(x)}{\lim_{x\to p} g(x)}$, assuming $g(x) \neq 0$ on some open interval containing p and $\lim_{x\to p} g(x) \neq 0$.

Remarks.

- The theorem applies to limits at $+\infty$ or $-\infty$ and to one-sided limits.
- The theorem easily follows from the corresponding properties of the limit of a sequence.

Corollary. (Properties of the limit ctd.)

1. $\lim_{x \to p}(f - g)(x) = \lim_{x \to p} f(x) - \lim_{x \to p} g(x)$,
2. $\lim_{x \to p}(c \cdot g)(x) = c \cdot \lim_{x \to p} g(x)$ for every constant c

Remark.

Part 2 follows from part 2 of the theorem with $f(x) = c$ for every x.
Part 1:

$\lim_{x \to p}(f - g)(x) = \lim_{x \to p}(f + (-1)g)(x) = \lim_{x \to p} f(x) + \lim_{x \to p}(-1)g(x) = \lim_{x \to p} f(x) - \lim_{x \to p} g(x)$.

Remark.

Since for every number c, $\lim_{x \to c} x = c$ we obtain that for every polynomial $f(x)$, $\lim_{x \to c} f(x) = f(c)$.

Theorem (Sandwich Theorem for functions)

Suppose functions f, g and h satisfy $f(x) \leq g(x) \leq h(x)$ on some open interval containing p and suppose $\lim_{x \to p} f(x)$ and $\lim_{x \to p} h(x)$ exist and both are equal to some number L. Then $\lim_{x \to p} g(x)$ exists and $\lim_{x \to p} g(x) = L$.

Remarks.

- The theorem applies to limits at $+\infty$ or $-\infty$ and to one-sided limits.
- The theorem follows from the sandwich theorem for sequences.

Definition.
The line $x = a$ is a *vertical asymptote* of a function f iff

$$\lim_{x\to a^+} f(x) = \pm\infty \text{ or } \lim_{x\to a^-} f(x) = \pm\infty$$

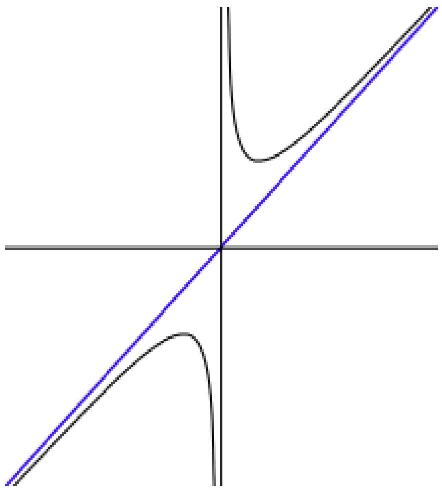
Definition.
The line $y = ax + b$ is an *oblique asymptote* of a function f iff

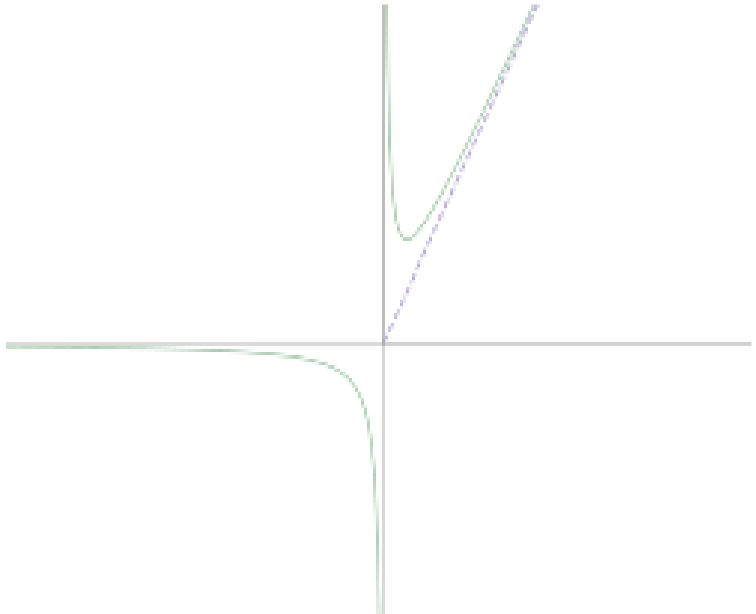
$$\lim_{x\to +\infty} [f(x) - (ax + b)] = 0 \text{ or } \lim_{x\to -\infty} [f(x) - (ax + b)] = 0.$$

When $a = 0$ the oblique asymptote is called the *horizontal asymptote*.

Essentially, we should say *asymptote* ***of the graph*** *of a function* because it is a geometrical object.

Example. *The graphs from Wikipedia*



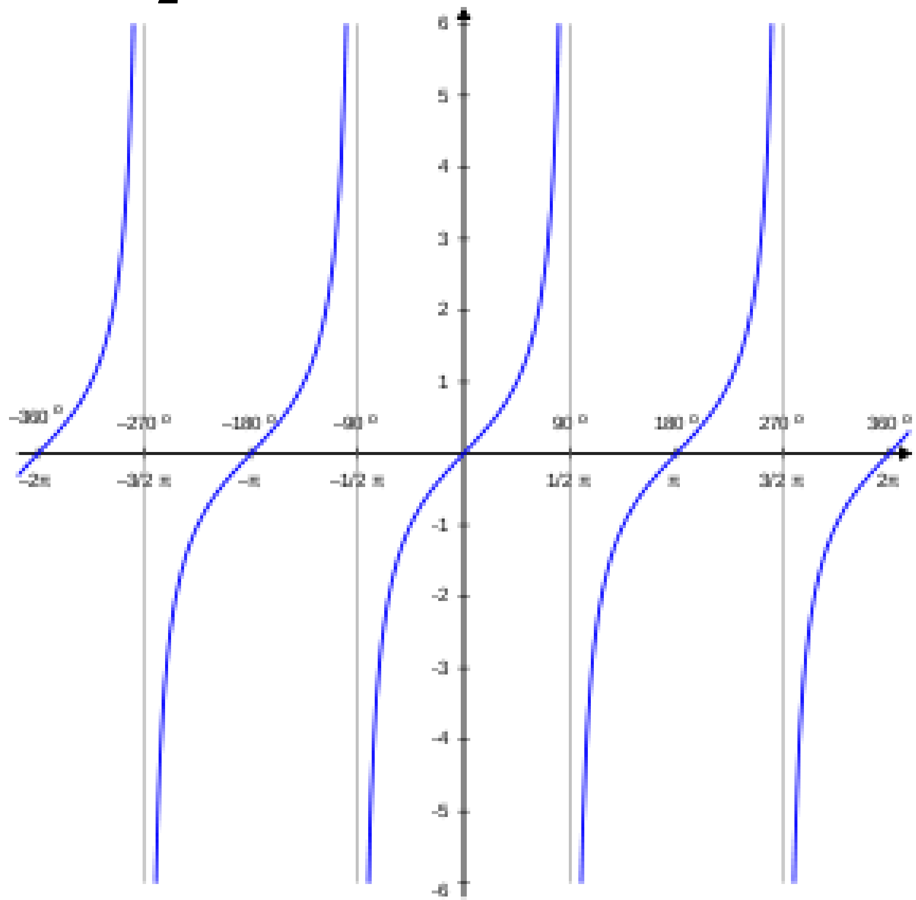


Lines $x = 0$ and $y = x$ are vertical and oblique asymptotes, respectively, for $f(x) = \frac{1}{x} + x$,

For this function, the line $y = 0$ is a horizontal asymptote at $-\infty$, line $x = 0$ is a vertical asymptote and $y = 2x$ is an oblique asymptote at $+\infty$.

Example. *The graph from Wikipedia*

Each line $y = k\pi + \frac{\pi}{2}$ is a vertical asymptote of (the graph of) tan



FAQ. How the hell do I find asymptotes for $f(x)$?

To find vertical asymptotes just look for points around which the values of your function are unbounded (division by zero, logarithm near zero and the like).

To find an oblique asymptote look at the definition: the line $y = ax + b$ is the oblique asymptote for $f(x)$ at $+\infty$ if and only if $\lim_{x\to+\infty}[f(x) - (ax+b)] = 0$. Dividing both sides by x we get $0 = \lim_{x\to+\infty} \frac{f(x)-(ax+b)}{x} = \lim_{x\to+\infty} \frac{f(x)}{x} - \lim_{x\to+\infty} \frac{ax}{x} - \lim_{x\to+\infty} \frac{b}{x} = \lim_{x\to+\infty} \frac{f(x)}{x} - a$ hence, $\lim_{x\to+\infty} \frac{f(x)}{x} = a$. Once we have a, we check if there exists the limit $\lim_{x\to+\infty}(f(x) - ax)$. If it does, then $b = \lim_{x\to+\infty}(f(x) - ax)$. If it does not – there is no asymptote at $+\infty$.

Theorem.

A function $f(x)$ has an asymptote at $+\infty$ iff there exist limits $\lim\limits_{x\to+\infty} \frac{f(x)}{x} = a$ and $\lim\limits_{x\to+\infty} (f(x) - ax) = b$. Then the line $y = ax+b$ is the asymptote.

Notice that the existence of $\lim\limits_{x\to+\infty} \frac{f(x)}{x}$ is not enough. For example consider $f(x) = \sqrt{x}$. $\lim\limits_{x\to+\infty} \frac{\sqrt{x}}{x} = 0 = a$ but $\lim\limits_{x\to+\infty} (\sqrt{x} - 0x)$ does not exist.

A similar theorem is valid for an oblique asymptote at $-\infty$.